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Exact Probability Distributions for Noncorrelated Random Walk Models

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A stochastic model for the idealized locomotion of cells is studied. The cell is assumed to cover a polygonal line in $\mathbb{R}^n$, the times between turns are exponentially distributed and independent of the directions, and the density of the nth direction e does not depend on the $(n-1)$th direction e'. The resulting Markov process $(X(t), D(t))$ for position and direction of the motion at time t is studied by using the integrodifferential equation for the transition function. For example, the joint distribution of $(X(t), D(t))$ is derived in closed form if $n=2$ or $n=3$ and all chosen directions (including the initial one) are uniformly distributed. For higher dimensions the combined Fourier–Laplace transform of $X(t)$ is given. The case of a fixed initial direction is also considered.

KEY WORDS: n-dimensional random walk; exact probability distribution.

1. INTRODUCTION

I consider a particle moving in n-dimensional space along straight-line paths which are separated by turns. While several authors consider models for planar motions of this type,[11,7,12,13] the analogous stochastic behavior in three dimensions is much less studied.[14,15] In this paper I present an n-dimensional model $(n \geqslant 2)$. Using this general approach, I am, for example, able to derive some new results for the two- and three-dimensional models. Random walks of this type are of frequent use in the description of cell motility.[1-10] I shall, however, focus on the case of noncorrelated directions in which explicit formulas for several densities and transforms turn out to be computable. The complexity of these calculations seems to indicate that in the case of nonuniform turn angle distributions (which are widely used in cell motion models) closed-form expressions cannot be expected.

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Let $X(t) \in \mathbb{R}^n$ be the position and $D(t) \in S^{n-1} = \{e \in \mathbb{R}^n \mid \|e\| = 1\}$ be the direction of motion of the particle at time t; $\|\cdot\|$ denotes Euclidean length.

The particle starts its motion at $X(0) = 0$ and covers its polygonal trajectory with unit speed. The initial direction $D(0)$ may have an arbitrary distribution. I make the following assumptions:

(i) The step lengths (i.e., the times between turns) are independent random variables with a common exponential distribution of mean $1/\lambda$ (this assumption can be substantiated by experimental observation of bacteria).

(ii) The sequence of chosen directions is independent of the sequence of step lengths.

(iii) Given that the nth direction is e', the $(n+1)$th direction e has a density depending only on the angle between e and e', say $\gamma(\langle e, e' \rangle)$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product.

Then the pair $(X(t), D(t))$ forms a Markov process with values in $\mathbb{R}^n \times S^{n-1}$. In Section 2 I derive the Fokker–Planck equation of this process, which will be the starting point of most of the subsequent results. I denote by $p(x, e; t)$ the probability density of $(X(t), D(t))$, i.e., for all Borel sets $B \subset \mathbb{R}^n$ and $C \subset S^{n-1}$,

$$\int_B \int_C p(x, e; t)\, dx\, dO(e) = P(X(t) \in B, D(t) \in C) \tag{1.1}$$

where O is the surface measure on S^{n-1}. In Section 3 a series expansion of $p(x, e; t)$ is derived.

The situation when all directions are uniformly distributed on S^{n-1} [i.e., $\gamma \equiv \text{const}$ and $D(0)$ is uniform] is amenable for explicit analysis, and is thus treated in this paper. In the case $n = 2$ we obtain

$$p(x, e; t) = \lambda \exp\{\lambda(t^2 - \|x\|^2)^{1/2} - \lambda t\}/4\pi^2(t - \langle x, e \rangle), \qquad 0 \leqslant \|x\| < t \tag{1.2}$$

The marginal distribution of $X(t)$ in the two-dimensional case has already been derived in ref. 12. The much more complicated three-dimensional case is settled in Section 4. If $q(x; t)$ is the density of $X(t)$, the following formula holds for $n = 3$:

$$q(x; t) = \frac{\lambda e^{-\lambda t}}{4\pi \|x\|} \left(\lambda \int_{-1}^{-\|x\|/t} \exp\{\lambda(tv + \|x\|) \operatorname{arctanh} v\} (\operatorname{arctanh} v)^2\, dv + \frac{1}{t} \operatorname{arctanh} \frac{\|x\|}{t} \right), \qquad 0 < \|x\| < t \tag{1.3}$$

The joint density of $X(t)$ and $D(t)$ can be derived from (1.3) by a formula which will be established in Section 2.

In Section 4 we assume that $\gamma \equiv \text{const}$ and the initial direction $D(0)$ has the density $h(e)$. In this case we obtain the combined Fourier–Laplace transform

$$\hat{p}(\theta, e; s) = \int_{\mathbb{R}^n} \int_0^\infty e^{i\langle \theta, x\rangle - st} p(x, e; t)\, dt\, dx, \qquad \theta \in \mathbb{R}^n, \quad s > 0 \tag{1.4}$$

of $p(x, e; t)$ with respect to x and t for fixed $e \in S^{n-1}$. It is given by

$$\begin{aligned} &\hat{p}(\theta, e; s) \\ &\quad = \frac{1}{s - i\langle e, \theta\rangle + \lambda} \\ &\qquad \times \left[h(e) + \frac{[\Gamma(n/2)\, \lambda/2\pi^{n/2}] \int_{S^{n-1}} [h(e)/(s - i\langle e, \theta\rangle + \lambda)]\, dO(e)}{1 - [\Gamma(n/2)\, \lambda/2\pi^{n/2}] \int_{S^{n-1}} [1/(s - i\langle e, \theta\rangle + \lambda)]\, dO(e)} \right] \end{aligned} \tag{1.5}$$

If the first direction is also uniformly distributed, we find that the Laplace transform of the characteristic function of $X(t)$ is given by

$$\begin{aligned} \hat{q}(\theta; s) &= \int_0^\infty E(e^{i\langle \theta, X(t)\rangle})\, e^{-st}\, dt \\ &= \frac{\arctan[\|\theta\|/(s+\lambda)]}{\|\theta\| - \lambda \arctan[\|\theta\|/(s+\lambda)]} \qquad \text{if} \quad n = 3 \end{aligned} \tag{1.6}$$

For $n \geqslant 4$ we also obtain closed-form expressions for $\hat{q}(\theta; s)$.

Particularly interesting for applications is the case of a fixed initial direction $D(0) = e_0$. Let $q(x; t \mid e_0)$ be the density of $X(t)$, given that $D(0) = e_0$. We show that

$$q(x; t \mid e_0) = O(S^{n-1})\, p_1(x, e_0; t) \tag{1.7}$$

where $p_1(x, e; t)$ is the density of $(X(t), D(t))$ under the condition $h \equiv 1/O(S^{n-1})$. By (1.7), the calculation of the density of $X(t)$ under a fixed initial direction is reduced to that of the density of $(X(t), D(t)$ for a uniform initial direction. Especially if $n = 2$ and $n = 3$, explicit expressions for $q(x; t \mid e_0)$ can be derived in this way.

2. THE INTEGRODIFFERENTIAL EQUATION FOR THE JOINT DENSITY

Although explicit results will be gained only in the case of non-correlated directions, I will present the n-dimensional model equations for the general case in this section. I retain the notation and the assumptions introduced in Section 1. The transition function $p(x', e'; t \mid x, e, s)$ of the Markov process $(X(t), D(t))$ has the properties

$$P(X(t) \in B, D(t) = e \mid X(s) = x, D(s) = e)$$

$$= \int_B p(x', e; t \mid x, e; s) \, dx'$$

$$\text{for all} \quad e \in S^{n-1}, \quad x \in \mathbb{R}^n, \quad B \subset \mathbb{R}^n, \quad 0 \leqslant s < t \tag{2.1}$$

$$P(X(t) \in B, D(t) \in C \mid X(s) = x, D(s) = e)$$

$$= \int_C \int_B p(x', e'; t \mid x, e; s) \, dx' \, dO(e)$$

$$\text{for all} \quad e \in S^{n-1}, \quad x \in \mathbb{R}^n, \quad B \subset \mathbb{R}^n, \quad C \subset S^{n-1} \setminus \{e\}, \quad 0 \leqslant s < t \tag{2.2}$$

where B and C are, of course, Borel sets and O is the usual surface measure on S^{n-1}. Relations (2.1) and (2.2) give the transition probabilities of the process corresponding to the cases of no change or at least one change of direction during the time interval $[s, t)$, respectively.

The Fokker–Planck equation for this transition function has the form

$$\frac{\partial}{\partial t} p(x', e'; t \mid x, e; s) + \langle e', \operatorname{grad}_{x'} p(x', e'; t \mid x, e; s) \rangle$$

$$+ \lambda p(x', e'; t \mid x, e; s)$$

$$= \lambda \int_{S^{n-1}} \gamma(\langle e', e'' \rangle) \, p(x', e'; t \mid x, e; s) \, dO(e'') \tag{2.3}$$

The proof of (2.3) follows well-known lines (for a related derivation see ref. 15) and will only be sketched. It is convenient to consider, for $C \subset S^{n-1}$,

$$p(x', C; t \mid x, e; s) = p(x', e; t \mid x, e; s) \, 1_C(e)$$

$$+ \int_C p(x', e'; t \mid x, e; s) \, dO(e') \tag{2.4}$$

where 1_C denotes the indicator function of the set C. If $h: \mathbb{R}^n \to \mathbb{R}$ is an arbitrary smooth function, one obtains

$$\int_{\mathbb{R}^n} h(x') \frac{\partial}{\partial t} p(x', C; t \mid x, e; s)\, dx'$$
$$= \lim_{\varepsilon \downarrow 0} \left(\varepsilon^{-1} \int_{\mathbb{R}^n} \int_{S^{n-1}} \left\{ \int_{\mathbb{R}^n} [h(x'') + \langle \operatorname{grad} h(x''), x' - x'' \rangle \right.\right.$$
$$+ O(\|x' - x''\|^2)]$$
$$\left. \times p(x', C; t+\varepsilon \mid x'', e'; t)\, dx' \right\} p(x'', de'; t \mid x, e; s)\, dx''$$
$$\left. - \varepsilon^{-1} \int_{\mathbb{R}^n} \int_{S^{n-1}} h(x')\, 1_C(e')\, p(x', de'; t \mid x, e; s)\, dx' \right) \tag{2.5}$$

Here we have used a Taylor expansion of h around x'' and the Chapman–Kolmogorov equation for $p(x', C; t \mid x, e; s)$. Under our assumptions we have

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} [P(D(t+\varepsilon) = e \mid X(t) = x, D(t) = e) - 1]$$
$$= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} [e^{-\lambda\varepsilon} - 1] = -\lambda \tag{2.6}$$

by the lack-of-memory property of the exponential waiting-time distribution between turns, and, if $e \notin C$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} P(D(t+\varepsilon) \in C \mid X(t) = x, D(t) = e)$$
$$= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} [\lambda\varepsilon e^{-\lambda\varepsilon} + o(\varepsilon)] \int_C \gamma(\langle e, e' \rangle)\, dO(e')$$
$$= \lambda \int_C \gamma(\langle e, e' \rangle)\, dO(e') \tag{2.7}$$

because for a Poisson process the probability of one event between t and $t+\varepsilon$ is equal to $\lambda\varepsilon e^{-\lambda\varepsilon}$ and that of at least two such events is of order $o(\varepsilon)$. Using (2.6), (2.7), and partial integration in (2.5) yields

$$\int_{\mathbb{R}^n} h(x') \frac{\partial}{\partial t} p(x', C; t \mid x, e; s)\, dx'$$
$$= \lambda \int_{\mathbb{R}^n} h(x'') \left[\int_C \int_{S^{n-1}} \gamma(\langle e', e'' \rangle)\, p(x'', de'; t \mid x, e; s)\, dO(e'') \right.$$
$$\left. - p(x'', C; t \mid x, e; s) \right] dx''$$
$$- \int_{\mathbb{R}^n} h(x'') \int_C \langle e', \operatorname{grad}_{x''} p(x'', de'; t \mid x, e; s) \rangle\, dx'' \tag{2.8}$$

Since h and C are arbitrarily chosen, (2.3) follows from (2.8).

Expression (2.3) is the fundamental equation for the transition function of the Markov process $(X(t), D(t))$ from which further information in special cases can be obtained. The analogous equation for the joint probability density $p(x, e; t)$ of $(X(t), D(t))$ is obviously given by

$$\frac{\partial}{\partial t} p(x, e; t) + \langle e, \operatorname{grad}_x p(x, e; t)\rangle + \lambda p(x, e; t)$$
$$= \lambda \int_{S^{n-1}} \gamma(\langle e, e'\rangle)\, p(x, e'; t)\, dO(e') \tag{2.9}$$

3. RECURSIVE RELATIONS FOR THE DENSITY OF $(X(t), D(t))$

I denote by $p_j(x, e'; t \mid e)$ the conditional joint density of $(X(t), D(t))$ given that the walk starts at the origin with the initial direction e and that there are j turns in the time interval $(0, t)$. The number of turns in $(0, t)$ has a Poisson distribution with parameter λ. By the formula of total probability, we have the series expansion

$$p(x, e'; t \mid e) = \sum_{j=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} p_j(x, e'; t \mid e) \tag{3.1}$$

by conditioning with respect to the number of turns in $(0, t)$. Obviously,

$$p_0(x, e'; t \mid e) = \delta(x - te)\, \delta(e' - e) \tag{3.2}$$

where $\delta(\cdot)$ denotes the Dirac delta function. I now establish the following recursive relation for the functions $p_j(x, e'; t \mid e)$:

$$p_j(x, e'; t \mid e)$$
$$= jt^{-j} \int_{S^{n-1}} \left\{ \int_{(\|x - te'\|^2)/(2(t - \langle x, e'\rangle))}^{t} u^{j-1} p_{j-1}(x - (t-u)\, e', e''; u \mid e)\, du \right\}$$
$$\times \gamma(\langle e', e''\rangle)\, dO(e'') \tag{3.3}$$

(3.3) is derived by conditioning with respect to the time U of the last turn in $(0, t)$. Given that there are j turns in $(0, t)$, their times have the same joint distribution as j independent random variables with the uniform distribution on $(0, t)$. Thus, U has the density $jt^{-j}u^{j-1}$ on $(0, t)$. One must have $X(u) = x - (t-u)\, e'$ in order to ensure that $X(t) = x$, and $\|x - (t-u)\, e'\| \leqslant u$ in order to ensure that $x - (t-u)\, e'$ is a possible displacement at time u. This latter condition is equivalent to

$u \geqslant \|x - te'\|^2/2(t - \langle x, e' \rangle)$. Given that $U = u$, the times of the first $j-1$ turns are distributed uniformly and independently on $(0, u)$. Thus, one can compute $p_j(x, e'; t \mid e)$ by integrating $p_{j-1}(x - (t-u)\, e', e''; u \mid e)\, \gamma(\langle e', e'' \rangle)$ with respect to e' and u, using the surface measure on S^{n-1} for e'' and the conditional density of U for u.

The series expansion (3.1) and (3.3) determine $p(x, e'; t \mid e)$ in principle. However, the occurring integrals seem difficult to handle. For the rest of the paper I consider the case of uniformly distributed changes of directions in more detail. Thus, let $\gamma(\langle e, e' \rangle) = 1/O(S^{n-1}) = \Gamma(n/2)/2\pi^{n/2}$ (the initial direction may be chosen according to an arbitrary distribution). Integrating (3.3) with respect to e, we obtain for the density $p_j(x, e'; t)$ of $(X(t), D(t))$, given that there are j turns in $(0, t)$, the recursion

$$p_j(x, e'; t) = \frac{\Gamma(n/2)\, j}{2\pi^{n/2} t^j} \int_{(\|x - te'\|^2)/(2(t - \langle x, e' \rangle))}^{t} u^{j-1} q_{j-1}(x - (t-u)\, e'; u)\, du \tag{3.4}$$

where

$$q_j(x; t) = \int_{S^{n-1}} p_j(x, e; t)\, dO(e) \tag{3.5}$$

Obviously,

$$q(x; t) = e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!}\, q_j(x; t) \tag{3.6}$$

is the density of $X(t)$. Expressions (3.4) and (3.6) imply the relation

$$p(x, e; t) = \frac{\Gamma(n/2)}{2\pi^{n/2}}\, \lambda \int_{(\|x - te\|^2)/(2(t - \langle x, e \rangle))}^{t} e^{-\lambda(t-u)} q(x - (t-u)\, e; u)\, du \quad \text{if} \quad x \neq te \tag{3.7}$$

If $x = te$, we also have $X(t) = x$ if no turns have taken place in $(0, t)$ and $D(0) = e$. Letting $h(e)$ denote the density of $D(0)$, we thus have to add $e^{-\lambda t} h(e)$ to the right-hand side of (3.7) in the case $x = te$. Equation (3.7) allows us to compute $p(x, e; t)$ if $q(x; t)$ is known.

Let us now further specialize to the case when also $D(0)$ is uniformly distributed. Then $q_0(x; t)$ corresponds to the uniform distribution on tS^{n-1}. Approximating this distribution by the uniform distribution on $t - k^{-1} < \|x\| < t + k^{-1}$, which has the density

$$(k\Gamma(n/2+1)/2n\pi^{n/2} t^{n-1})\, 1_{(t-k^{-1}, t+k^{-1})}(\|x\|)$$

we can compute $p_1(x, e; t)$ as follows:

$$p_1(x, e; t) = \lim_{k \to \infty} \frac{\Gamma(n/2)}{2\pi^{n/2} t} \int_{(\|x - te\|^2)/(2(t - \langle x, e \rangle))}^{t} \frac{k\Gamma(n/2+1)}{2n\pi^{n/2} u^{n-1}} \times 1_{(u-k^{-1}, u+k^{-1})}(\|x - (t-u)\, e'\|)\, du$$

$$= \frac{\Gamma(n/2)^2}{8\pi^n t} \lim_{k \to \infty} k \int_{(\|x - te\|^2)/(2(t - \langle x, e \rangle))}^{(\|x - te\|^2)/(2(t - \langle x, e \rangle - 2k^{-1}))} u^{1-n}\, du \quad (3.8)$$

In the two-dimensional case ($n = 2$) we obtain

$$p_1(x, e; t) = \frac{1}{4\pi^2 t(t - \langle x, e \rangle)} \quad (3.9)$$

Thus,

$$q_1(x; t) = \int_{S^1} \frac{dO(e)}{4\pi^2 t(t - \langle x, e \rangle)} = \frac{1}{2\pi t^2 [1 - (\|x\|/t)^2]^{1/2}} \quad (3.10)$$

Now it follows easily from (3.4) that

$$p_j(x, e; t) = j[1 - (\|x\|/t)^2]^{(j-1)/2} / 4\pi^2 t(t - \langle x, e \rangle), \qquad j \geqslant 1 \quad (3.11)$$

Inserting (3.11) into (3.1) yields

$$p(x, e; t) = \lambda \exp\{\lambda(t^2 - \|x\|^2)^{1/2} - \lambda t\} / 4\pi^2 (t - \langle x, e \rangle), \qquad 0 \leqslant \|x\| < t \quad (3.12)$$

Except for a constant factor, the joint density of $(X(t), D(t))$ is thus the product of $\exp\{\lambda(t^2 - \|x\|^2)^{1/2}\}$, a monotone decreasing function of $\|x\|$, and of the function $(t - \langle x, e \rangle)^{-1}$, reflecting the dependence of the joint density on the angle between x and e. If we fix the cosine, say β, between x and e, the density becomes, except for a constant factor,

$$f_\beta(u) = (t - \beta u)^{-1} \exp\{\lambda(t^2 - u^2)^{1/2}\}$$

where we have set $u = \|x\| < t$. If $\beta \leqslant 0$, this function decreases monotonically on $[0, t)$. If $0 < \beta < 1$, there is an $u_0 \in (0, t)$ such that f increases on $[0, u_0]$ and decreases on $[u_0, t)$. The u_0 can be determined as the unique solution of the equation $\lambda u(t - \beta u) = \beta(t^2 - u^2)^{1/2}$ in the interval $[0, t)$. Only in the case $\beta = 1$ (i.e., $x/\|x\| = e$) do we have $\lim_{u \uparrow t} f(u) = \infty$. Regarding monotonicity, for $\beta = 1$ two cases are possible. Let

$$c_t = \inf_{0 < u < t} u^{-1} \left(\frac{t+u}{t-u}\right)^{1/2} = \frac{c_1}{t} = \frac{2}{5^{1/2} - 1} \left(\frac{5^{1/2} + 1}{3 - 5^{1/2}}\right)^{1/2} \frac{1}{t} = 3.33019 \cdots \frac{1}{t}$$

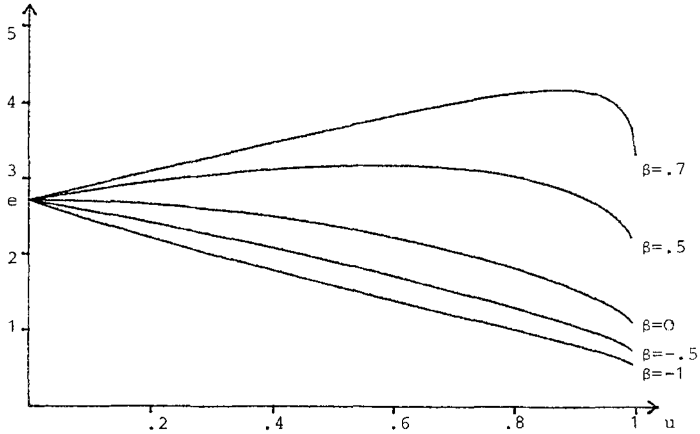

Fig. 1. Plot of $f_\beta(u)$ for $t=\lambda=1$ and different values of β (= −1, 0.5, 0, 0.5, 0.7).

If $\lambda \leqslant c_t$, f_1 is monotone increasing on $[0, t)$. If $\lambda > c_t$, the equation $\lambda u(t-u) = (t^2-u^2)^{1/2}$ has exactly two solutions u_1, $u_2 \in (0, t)$, $u_1 < u_2$, and f_1 is monotone increasing on $[0, u_1]$ and on $[u_2, t)$ and monotone decreasing on $[u_1, u_2]$.

Figure 1 shows the graph of $f_\beta(u)$ for $t=\lambda=1$ and different values of β ($\beta = -1, -0.5, 0, 0.5, 0.7$). Figure 2 gives the graph of $e^{-\lambda} f_1(u)$ for $t=1$ and $\lambda = 1, 2, 3, 4, 7$.

From (3.12) the marginal and conditional densities of $X(t)$ and $D(t)$

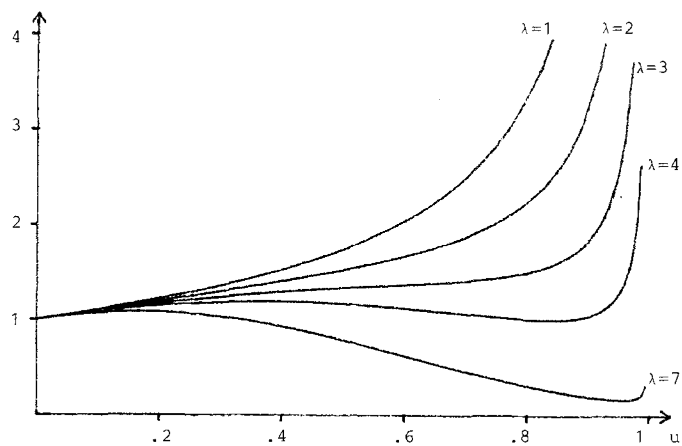

Fig. 2. Plot of $f_1(u)$ for $t=1$ and different values of λ (= 1, 2, 3, 4, 7).

can be derived. $D(t)$ is obviously uniformly distributed on S^1, and for the density $q(x; t)$ of $X(t)$, we obtain

$$q(x: t) = \frac{\lambda}{2\pi} (t^2 - \|x\|^2)^{-1/2} \exp\{\lambda(t^2 - \|x\|^2)^{1/2} - \lambda t\} \tag{3.13}$$

(3.13) has already been derived and discussed in ref. 12.

The conditional density of $D(t)$, given that $X(t) = x$, is given by

$$p_t(e \mid x) = \frac{(t^2 - \|x\|^2)^{1/2}}{2\pi(t - \langle x, e \rangle)}, \qquad e \in S^1, \quad \|x\| < t \tag{3.14}$$

since $p(x, e; t) = q(x; t)\, p_t(e \mid x)$.

Thus, if we denote by β the angle between $D(t)$ and $X(t)$, β has, under the condition $\|X(t)\|/t = b$ (for fixed $b \in [0, 1)$), the density

$$g_t(\beta \mid b) = \frac{(1 - b^2)^{1/2}}{2\pi(1 - b \cos \beta)}, \qquad \beta \in [0, 2\pi)$$

This density is symmetric around π, decreasing on $[0, \pi]$, and increasing on $[\pi, 2\pi)$. If b is small, it is approximately constant; if $b \uparrow 1$, it tends to be concentrated at 0 and 2π. Thus, if $\|x\|$ is small compared with t, the conditional distribution $v_{t,x}$ of $D(t)$, given that $X(t) = x$, is nearly uniform; if $\|x\|/t \uparrow 1$, $v_{t,x}$ converges weakly to the point mass at $x/\|x\|$. In general, $v_{t,x}$ has a density with a maximum at $x/\|x\|$, decreasing from it in both directions (clockwise and counterclockwise) symmetrically to its minimum at $-x/\|x\|$.

If $n \geqslant 3$, the situation becomes more difficult. The calculation of the limit on the right-hand side of (3.8) gives

$$p_1(x, e; t) = \frac{2^{n-4}\Gamma(n/2)^2}{\pi^n t} \frac{(t - \langle x, e \rangle)^{n-3}}{\|x - te\|^2}, \qquad n \geqslant 3 \tag{3.15}$$

For example, if $n = 3$,

$$\begin{aligned} q_1(x; t) &= (8\pi^2 t)^{-1} \int_{S^2} \|x - te\|^{-2}\, dO(e) \\ &= (8\pi^2 t)^{-1} \int_0^{2\pi} \int_0^{\pi} \frac{\sin v_1\, dv_1\, dv_2}{\|x\|^2 + t^2 - 2\, \|x\|\, t \cos v_1} \\ &= (4\pi^2 t^2\, \|x\|)^{-1} \left[(\|x\| - t)^{-4} - (\|x\| + t)^{-4}\right] \end{aligned} \tag{3.16}$$

so that, by (3.4),

$$p_2(x, e; t) = (32\pi^3 t^2)^{-1} \int_{(\|x-te\|^2)/(2(t-\langle x,e\rangle))}^{t} u^{-1} \|x-(t-u)e\|^{-1} \times \{[\|x-(t-u)e\| - u]^{-4} - [\|x-(t-u)e\| + u]^{-4}\} \, du \tag{3.17}$$

This integral (as well as the corresponding ones for larger values of j) apparently cannot be given in closed form. In the next section we shall explicitly derive $q(x; t)$ in the three-dimensional case by combining the approach developed above and the use of (2.9).

4. EXPONENTIAL STEP LENGTHS AND UNIFORM DIRECTIONS IN THE THREE-DIMENSIONAL CASE

Let us now study the motion of a particle starting at the origin of the (x_1, x_2, x_3) space and choosing independently exponentially distributed step lengths (with mean $1/\lambda$) and directions which are uniformly distributed on the sphere S^2. Let $q(x; t)$ be the density of the displacement at time t; obviously, q depends on x only through $\|x\|$, so that we can write $q(x; t) = \tilde{q}(\|x\|; t)$, $x \in \mathbb{R}^3$.

The approach developed in Section 3 apparently does not lead to a closed-form expression for $q(x; t)$. Therefore we have to invent a little trick and consider the following related problem. Assume that the starting point is distributed uniformly in the plane $x_3 = 0$ [i.e., take the Lebesgue measure in the (x_1, x_2) plane as prior density]; after the starting point is determined, the motion proceeds as is described above. Let $r(x_1, x_2, x_3; t)$ be the density at time t. Obviously,

$$\begin{aligned} r(x_1, x_2, x_3; t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{q}(\|(x_1 - x_1', x_2 - x_2', x_3)\|; t) \, dx_1' \, dx_2' \\ &= 2\pi \int_0^{\infty} \tilde{q}((y^2 + x_3^2)^{1/2}; t) \, y \, dy \\ &= 2\pi \int_{|x_3|}^{\infty} \tilde{q}(v; t) \, v \, dv \end{aligned} \tag{4.1}$$

Let $\tilde{r}(u; t) = r(x_1, x_2, u; t)$ for $u \in \mathbb{R}$. Then, by (4.1),

$$\tilde{q}(u; t) = -(2\pi u)^{-1} \frac{\partial}{\partial u} \tilde{r}(u; t), \qquad 0 < u < t \tag{4.2}$$

Thus it suffices to compute $\tilde{r}(u; t)$. Let $p(x_1, x_2, x_3, e; t)$ $(e \in S^2)$ be the joint density of displacement and direction at time t. Clearly, p depends on e only through the cosine, say α, between e and the $x_3$axis; further, p is independent of x_1 and x_2. Therefore the relations between $\tilde{p}(u, \alpha; t) = p(0, 0, u, e; t)$ and $\tilde{r}(u; t)$ are, by (2.9), given by

$$\frac{\partial}{\partial t} \tilde{p}(u, \alpha; t) + \alpha \frac{\partial}{\partial u} \tilde{p}(u, \alpha; t) + \lambda \tilde{p}(u, \alpha; t) = (\lambda/4\pi)\, \tilde{r}(u; t) \tag{4.3}$$

(4π is the surface measure of S^2) and, by a change to polar coordinates,

$$\begin{aligned} \tilde{r}(u; t) &= \int_{S^2} p(x_1, x_2, u, e; t)\, dO(e) \\ &= 2\pi \int_{-1}^{1} \tilde{p}(u, \alpha; t)\, d\alpha \end{aligned} \tag{4.4}$$

The initial condition for $\tilde{p}$ is given by

$$\tilde{p}(u, \alpha; 0) = \delta(u)/4\pi \tag{4.5}$$

Analogously as in (3.6), we can write

$$\tilde{r}(u; t) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \tilde{r}_n(u; t) \tag{4.6}$$

and

$$\tilde{p}(u, \alpha; t) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \tilde{p}_n(u, \alpha; t) \tag{4.7}$$

where $\tilde{r}_n$ is the conditional density of the displacement in the third coordinate, given that there are n turns up to time t, and $\tilde{p}_n$ is defined similarly.

Moreover, we have

$$\tilde{r}_n(u; t) = t^{-1} \tilde{r}_n(u/t; 1) \tag{4.8}$$

$$\tilde{p}(u, \alpha; t) = t^{-1} \tilde{p}_n(u/t, \alpha; 1) \tag{4.9}$$

To see (4.8), note that the left-hand side is the integral with respect to the first two coordinates (x_1, x_2) of the density of a sum of the form $(x_1, x_2, 0) + S$, where S is a sum of $n+1$ independent, rotationally invariant random vectors whose lengths are given by the $n+1$ spacings of a sample of size n from the rectangular distribution on $(0, t)$. The right-hand side can be interpreted in the same way except that S must be replaced by tS' and the interval $(0, t)$ by $(0, 1)$. Obviously, S and tS' have the same distribution. (4.9) can be derived by a similar argument.

Inserting (4.8) and (4.9) into (4.6) and (4.7) and inserting the resulting series into (4.3) and (4.4) yields for $r_n(u) = \tilde{r}_n(u; 1)$ and $p_n(u, \alpha) = 4\pi\tilde{p}_n(u, \alpha; 1)$ the equations

$$\sum_{n=0}^{\infty} \frac{\lambda^n t^{n-2}}{n!} \left[(n-1)\, p_n(y, \alpha) + (\alpha - y) \frac{\partial}{\partial y} p_n(y, \alpha) \right] = \sum_{n=0}^{\infty} \frac{\lambda^{n+1} t^{n-1}}{n!} r_n(y) \tag{4.10}$$

where $y = u/t$, and

$$\sum_{n=0}^{\infty} \frac{\lambda^n t^{n-1}}{n!} \frac{1}{2} \int_{-1}^{1} p_n(y, \alpha)\, d\alpha = \sum_{n=0}^{\infty} \frac{\lambda^n t^{n-1}}{n!} r_n(y) \tag{4.11}$$

Comparing coefficients in (4.10) and (4.11), we obtain

$$(\alpha - u) \frac{\partial}{\partial u} p_n(u; \alpha) + (n-1)\, p_n(u; \alpha) = n r_{n-1}(u) \tag{4.12}$$

$$\frac{1}{2} \int_{-1}^{1} p_n(u; \alpha)\, d\alpha = r_n(u) \tag{4.13}$$

Differentiating (4.12) $n-1$ times gives

$$\frac{\partial^n}{\partial u^n} p_n(u; \alpha) = \frac{n}{\alpha - u} \frac{d^{n-1}}{du^{n-1}} r_{n-1}(u) \tag{4.14}$$

Integrate (4.14) with respect to α and use (4.13) to obtain

$$\frac{d^n}{du^n} r_n(u) = -n \operatorname{arctanh} u \frac{d^{n-1}}{du^{n-1}} r_{n-1}(u) \tag{4.15}$$

{note that $\operatorname{arctanh} u = \frac{1}{2}\log[(u+1)/(1-u)]$}. Since $r_0(u) = 1/2$, $-1 \leqslant u \leqslant 1$, (4.15) implies that

$$\frac{d^n}{du^n} r_n(u) = (-1)^n \frac{n!}{2} (\operatorname{arctanh} u)^n \tag{4.16}$$

By Taylor's formula,

$$r_n(u) = \sum_{i=0}^{n-1} \frac{1}{i!} (u+1)^i \frac{d^i}{du^i} r_n(-1) + (-1)^n \frac{n}{2} \int_{-1}^{u} (u-x)^{n-1} (\operatorname{arctanh} x)^n\, dx, \qquad u \in [-1, 0] \tag{4.17}$$

The polynomial term on the right-hand side of (4.17) vanishes. We have

$$\frac{d^i}{du^i} r_n(-1) = 0 \qquad \text{for all} \quad n \geqslant 1 \text{ and } i = 0, 1, 2, \ldots, n-1 \tag{4.18}$$

The proof of (4.18) turns out to require laborious calculations (which present the main difficulty of this derivation) and is given in the Appendix. Taking (4.18) for granted, (4.17) gives for $r_n(u)$ the identity

$$r_n(u) = (-1)^n \frac{n}{2} \int_{-1}^{u} (u-x)^{n-1} (\operatorname{arctanh} x)^n \, dx, \qquad u \in [-1, 0], \quad n \geqslant 1 \tag{4.19}$$

Now we use (4.8), (4.6), and the symmetry property $r_n(u) = r_n(-u)$ to obtain

$$\begin{aligned} \tilde{r}(u; t) &= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} [\exp(-\lambda t)] \, t^{-1} r_n\left(\frac{u}{t}\right) \\ &= [\exp(-\lambda t)] \left\{ (2t)^{-1} - \frac{\lambda}{2} \int_{-1}^{-u/t} \exp[\lambda(tx+u) \operatorname{arctanh} x] \right. \\ &\quad \left. \times \operatorname{arctanh} x \, dx \right\}, \qquad 0 < u < t \end{aligned} \tag{4.20}$$

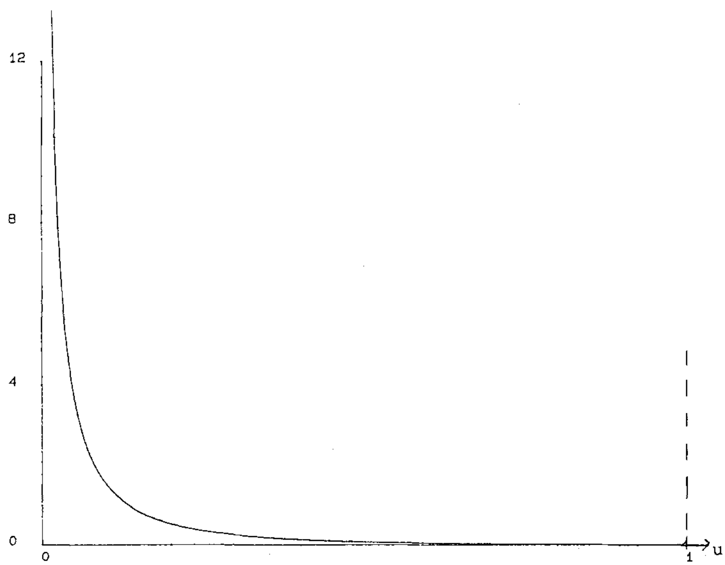

Fig. 3. Plot of $\tilde{q}(u; 1)$ for $\lambda = 3$.

By (4.2) we finally arrive at an expression for $\tilde{q}(u; t)$:

$$\tilde{q}(u; t) = \frac{\lambda e^{-\lambda t}}{4\pi u} \left(\lambda \int_{-1}^{-u/t} \exp\{\lambda(tx+u) \operatorname{arctanh} x\} \times (\operatorname{arctanh} x)^2 \, dx + \frac{1}{t} \operatorname{arctanh} \frac{u}{t} \right), \qquad 0 < u < t \tag{4.21}$$

$\tilde{q}(u; t)$ is of order $O(u^{-1})$, as $u \downarrow 0$, and of order $O(|\log(t-u)|)$, as $u \uparrow t$. The typical shape of $\tilde{q}(u; t)$ is shown in Fig. 3, where I have taken $t = 1$ and $\lambda = 3$. The increase to infinity as $u \uparrow 1$ is too slow to be represented in the plot.

5. TRANSFORMS

In the special case of exponential step lengths and uniformly distributed changes of directions one can obtain the combined Fourier–Laplace transform of $p(x, e; t)$ (with respect to x and t for fixed e) directly from (2.9). Thus, let again $\gamma(\langle e, e' \rangle) = 1/O(S^{n-1}) = \Gamma(n/2)/2\pi^{n/2}$. Then (2.9) takes the form

$$\frac{\partial}{\partial t} p(x, e; t) + \langle e, \operatorname{grad}_x p(x, e; t) \rangle + \lambda p(x, e; t) = \frac{\Gamma(n/2)\, \lambda}{2\pi^{n/2}} q(x; t) \tag{5.1}$$

where

$$q(x; t) = \int_{S^{n-1}} p(x, e; t)\, dO(e) \tag{5.2}$$

Obviously $q(x; t)$ is the probability density of $X(t)$, the displacement at time t. Let

$$\hat{p}(\theta, e; s) = \int_{\mathbb{R}^n} \int_0^\infty e^{i\langle \theta, x \rangle - st} p(x, e; t)\, dt\, dx, \qquad \theta \in \mathbb{R}^n, \quad s \geqslant 0 \tag{5.3}$$

$$\hat{q}(\theta; s) = \int_{\mathbb{R}^n} \int_0^\infty e^{i\langle \theta, x \rangle - st} q(x; t)\, dt\, dx = \int_0^\infty E(e^{i\langle \theta, X(t) \rangle})\, e^{-st}\, dt \tag{5.4}$$

Suppose that $X(0) = 0$ and the first direction has the probability density $h(e)$ with respect to $O(de)$.

Transforming both sides of (5.1) yields the identities

$$\hat{p}(\theta, e; s) = [s - i\langle e, \theta\rangle + \lambda]^{-1} \left[\frac{\Gamma(n/2)\,\lambda}{2\pi^{n/2}}\,\hat{q}(\theta, s) + h(e)\right] \tag{5.5}$$

$$\hat{q}(\theta; s) = \frac{\int_{S^{n-1}} [h(e)/(s - i\langle e, \theta\rangle + \lambda)]\, dO(e)}{1 - [\Gamma(n/2)\,\lambda/2\pi^{n/2}] \int_{S^{n-1}} [1/(s - i\langle e, \theta\rangle + \lambda)]\, dO(e)}$$

$$\theta \in \mathbb{R}^n, \qquad s > 0 \tag{5.6}$$

In particular, formula (1.5) follows.

As an application, let us consider the three-dimensional case with constant $h(e) \equiv 1/4\pi$ so that all directions (including the initial one) are uniformly distributed on the sphere S^2. Obviously, $\hat{q}(\theta; s)$ depends on θ only through $\|\theta\| = \rho$. Introducing polar coordinates, we obtain

$$\int_{S^2} (s - i\langle e, \theta\rangle + \lambda)^{-1}\, dO(e)$$

$$= \int_{S^2} (s + \lambda - i\rho e_1)^{-1}\, dO(e)$$

$$= 2\pi \int_0^\pi \frac{\sin t\, dt}{s + \lambda - i\rho\cos t} = \rho^{-1} \arctan\frac{\rho}{s + \lambda} \tag{5.7}$$

Thus,

$$\hat{q}(\theta; s) = \int_0^\infty E(e^{i\langle\theta, X(t)\rangle})\, e^{-st}\, dt$$

$$= \frac{(1/\|\theta\|) \arctan[\|\theta\|/(s + \lambda)]}{1 - (\lambda/\|\theta\|) \arctan[\|\theta\|/(s + \lambda)]} \tag{5.8}$$

By Laplace inversion of $\hat{q}(\theta; s)$, one can find the characteristic function of $X(t)$; however, there seems to be no closed-form expression for this function.

For general n we have to compute

$$\int_{S^{n-1}} (s - i\langle e, \theta\rangle + \lambda)^{-1}\, dO(e)$$

$$= 2\pi \prod_{k=3}^{n-1} \left(\int_0^\pi \sin^{n-k} v\, dv\right) \int_0^\pi \frac{\sin^{n-2} v}{s + \lambda - i\,\|\theta\|\cos v}\, dv \tag{5.9}$$

If $n=4$, this is equal to $4\pi^2 \|\theta\|^{-2} \{[(s+\lambda)^2+\|\theta\|^2]^{1/2}-s-\lambda\}$, so that, by (4.6),

$$\hat{q}(\theta; s)=\frac{2\{[(s+\lambda)^2+\|\theta\|^2]^{1/2}-s-\lambda\}}{\|\theta\|^2-2\lambda\{[(s+\lambda)^2+\|\theta\|^2]^{1/2}-s-\lambda\}} \tag{5.10}$$

If $n\geqslant 5$, we can use the formulas (see ref. 16, p. 378)

$$\int_0^\pi \sin^m v\, dv=\frac{\Gamma((m+1)/2)\,\Gamma(1/2)}{\Gamma((m+2)/2)}, \qquad m\geqslant 1 \tag{5.11}$$

$$\begin{aligned}\int_0^\pi &\frac{\sin^m v\, dv}{\alpha-i\rho\cos v}\\ &=\left(\frac{a^2+\rho^2}{4\rho^2}\right)^k A\\ &-\frac{2^{m-2}a}{\rho^2}\sum_{v=1}^{k}\left(\frac{a^2+\rho^2}{4\rho^2}\right)^{v-1} B\left(\frac{m+1-2v}{2},\frac{m+1-2v}{2}\right)\end{aligned} \tag{5.12}$$

where $m\geqslant 3$, $a>0$, $B(x, y)$ is the beta function, k is the largest integer $\leqslant (m-1)/2$, and

$$A=\frac{\pi a}{\rho^2}\left[\left(1+\frac{\rho^2}{a^2}\right)^{1/2}-1\right] \qquad \text{if} \quad m \text{ is even}$$

$$A=\frac{2}{\rho}\arctan\frac{\rho}{a}, \qquad \text{if} \quad m \text{ is odd}$$

Inserting (5.11) and (5.12) into (5.9) and the result into (5.6) gives the desired closed-form expression for $\hat{q}(\theta; s)$.

In applications the initial direction of the random walk is often assumed to be known. Let $q(x; t\,|\,e_0)$ be the density of $X(t)$, given that $D(0)=e_0$. Its transform $\hat{q}(\theta; s\,|\,e_0)$ is given by (5.6) if we replace $h(e)\, dO(e)$ by $\delta(e-e_0)$. One obtains

$$\hat{q}(\theta; s\,|\,e_0)=[s-i\langle e, \theta\rangle+\lambda]^{-1}\left(\frac{2\pi^{n/2}}{\Gamma(n/2)}-\int_{S^{n-1}}\frac{dO(e)}{s-i\langle e, \theta\rangle+\lambda}\right)^{-1} \tag{5.13}$$

Let us denote by $f(x, e; t)$ the joint density of $(X(t), D(t))$ in the case $h\equiv 1/O(S^{n-1})$, and let $\hat{f}(\theta, e; s)$ be its transform (5.3). Then, calculating $\hat{f}(\theta, e; s)$ from (5.5) and (5.6) and comparing the result with (5.13) shows that

$$\hat{q}(\theta; s\,|\,e_0)=\frac{2\pi^{n/2}}{\Gamma(n/2)}\hat{f}(\theta, e_0; s) \tag{5.14}$$

Thus we arrive at the following identity between densities:

$$q(x; t \mid e_0) = \frac{2\pi^{n/2}}{\Gamma(n/2)} f(x, e_0; t) \tag{5.15}$$

Equation (5.15) allows us to reduce the computation of the density of $X(t)$ for arbitrary fixed initial direction to that of the density of $(X(t), D(t))$ for uniformly distributed initial direction. For example, in the two-dimensional case we obtain from (3.12)

$$q(x; t \mid e_0) = \lambda \exp\{\lambda(t^2 - \|x\|^2)^{1/2} - \lambda t\}/2\pi(t - \langle x, e_0 \rangle)$$
$$0 \leqslant \|x\| < t, \qquad x \neq te_0 \tag{5.16}$$

In the three-dimensional case we can use (4.32), (3.7), and (5.15) to represent $q(x; t \mid e_0)$ as a complicated double integral:

$$q(x; t \mid e_0)$$
$$= \frac{\lambda^2}{4\pi} \int_{\|x - te_0\|^2/(2(t - \langle x, e_0 \rangle))}^{t} \{\exp[-\lambda(t-u)]\} \, \|x - (t-u)\, e_0\|^{-1}$$
$$\times \left[\int_{-1}^{-\|x - (t-u)e_0\|/u} \exp\{\lambda(uv + \|x - (t-u)\, e_0\|) \operatorname{arctanh} v\} \right.$$
$$\left. \times (\operatorname{arctanh} v)^2 \, dv + \frac{1}{u} \operatorname{arctanh} \frac{\|x - (t-u)\, e_0\|}{u} \right] du \tag{5.17}$$

if $x \neq te_0$ [at $x = te_0$ the distribution of $X(t)$ given $D(0) = e_0$ has of course an atom of size $e^{-\lambda t}$]. Its transform is, however, of a simple form:

$$\hat{q}(\theta, s \mid e_0) = \left[(s - i\langle e_0, \theta \rangle + \lambda) \left(1 - \frac{\lambda}{\|\theta\|} \arctan \frac{\|\theta\|}{s + \lambda} \right) \right]^{-1} \tag{5.18}$$

APPENDIX. PROOF OF FORMULA (4.18)

Let

$$\hat{p}_n(s, \alpha; t) = \int_{-\infty}^{\infty} e^{isu} \tilde{p}_n(u, \alpha; t)\, du$$

$$\hat{r}_n(s; t) = \int_{-\infty}^{\infty} e^{isu} \tilde{r}_n(u; t)\, du$$

be the Fourier transforms of $\tilde{p}_n$ and $\tilde{r}_n$ with respect to u. By (4.3), we have

$$\frac{\partial}{\partial t}\,\hat{p}(s,\alpha;t)+(i\alpha s+\lambda)\,\hat{p}(s,\alpha;t)=\frac{\lambda}{4\pi}\,\hat{r}(s;t) \tag{A.1}$$

Therefore $e^{\lambda t}\hat{p}(s,\alpha;t)$ satisfies, as a function of t, the linear differential equation

$$\frac{\partial}{\partial t}\,(e^{\lambda t}\hat{p}(s,\alpha;t))=-i\alpha s e^{\lambda t}\hat{p}(s,\alpha;t)+e^{\lambda t}\frac{\lambda}{4\pi}\,\hat{r}(s;t) \tag{A.2}$$

and is thus given by

$$e^{\lambda t}\hat{p}(s,\alpha;t)=e^{-i\alpha s t}\left[\int_0^t \frac{\lambda}{4\pi}\,\hat{r}(s;y)\,e^{\lambda y+i\alpha s y}\,dy+C\right] \tag{A.3}$$

Since $\hat{p}(s,\alpha;0)=1/4\pi$, the constant C is equal to $1/4\pi$. Integrating (A.3) with respect to α yields

$$\begin{aligned}2\pi\int_{-1}^{1} & e^{\lambda t}\hat{p}(s,\alpha;t)\,d\alpha\\ &=\lambda\int_0^t \hat{r}(s;y)\,e^{\lambda y}\,\frac{\sin s(y-t)}{s(y-t)}\,dy+\frac{\sin st}{st}\end{aligned} \tag{A.4}$$

The left-hand side of (A.4) is the Fourier transform of $e^{\lambda t}r(u;t)$, and the integrand at the right-hand side is the product of the Fourier transforms of $e^{\lambda t}r(u;y)$ and the uniform density $[2(t-y)]^{-1}\,1_{[-(t-y),t-y]}(u)$ (all with respect to u). This can be utilized as follows. Let $g(u,v)=\exp[\lambda(u+v)/2]\,\tilde{r}((u-v)/2;(u+v)/2)$. Taking inverse Fourier transforms in (A.4), it is readily calculated that g satisfies the integral equation

$$g(u,v)=(u+v)^{-1}+\frac{\lambda}{2}\int_0^u\left[\int_0^v \frac{g(x,z)}{u-x+v-z}\,dz\right]dx, \qquad u,v>0 \tag{A.5}$$

Indeed, by (A.4),

$$g(t+s,t-s)=\lambda\int_0^t \frac{1}{2(t-y)}\left[\int_{-(t-y)}^{t-y} g(y+s-z',\,y-(s-z'))\,dz'\right]dy+(2t)^{-1} \tag{A.6}$$

and the transformations $u=t+s$, $v=t-s$, $x=y+s-z'$, and $z=y-(s-z')$ prove (A.5) [note that $g(x,z)=0$ if $x+z<|x-z|$].

It follows from the definition of g, (4.6), and (4.8) that

$$g(u, v) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!\, 2^{n-1}} (u+v)^{n-1} r_n\left(\frac{u-v}{u+v}\right), \qquad u, v > 0 \tag{A.7}$$

and, by (A.5),

$$g(u, v) = (u+v)^{-1} + \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!\, 2^n} \int_0^u \int_0^v \frac{(x+z)^{n-1}}{u-x+v-z} \times r_n\left(\frac{x-z}{x+z}\right) dz\, dx, \qquad u, v > 0. \tag{A.8}$$

Equating the coefficients of λ^n in (A.7) and (A.8) shows that

$$\begin{aligned} r_n\left(\frac{u-v}{u+v}\right) &= n(u+v)^{-2} \int_0^u \int_0^v \frac{[(x+z)/(u+v)]^{n-2}}{1-[(x+z)/(u+v)]} r_{n-1}\left(\frac{x-z}{x+z}\right) dz\, dx \\ &= \frac{n}{2} \iint_D \frac{y^{n-1}}{1-y} r_{n-1}(w)\, dy\, dw \end{aligned} \tag{A.9}$$

where the second equation follows from the change of variables $y = (x+z)/(u+v)$, $w = (x-z)/(x+z)$. The region of integration D is given by

$$\begin{aligned} D &= \left\{ \left(\frac{x+z}{u+v}, \frac{x-z}{x+z} \right) \middle| x \in (0, u),\, z \in (0, v) \right\} \\ &= \left\{ (y, w) \middle| w \in \left(\frac{u-v}{u+v}, 1 \right),\, y \in \left(0, \frac{2u}{(u+v)(1+w)} \right) \right\} \\ &\quad \cup \left\{ (y, w) \middle| w \in \left(-1, \frac{u-v}{u+v} \right],\, y \in \left(0, \frac{2v}{(u+v)(1-w)} \right) \right\} \end{aligned} \tag{A.10}$$

For arbitrary $x \in [-1, 1]$ choose u, v such that $u+v=1$ and $x = u-v$. Then, by (A.9) and (A.10), we obtain the recursion

$$\begin{aligned} r_n(x) &= \frac{n}{2} \int_{-1}^{x} r_{n-1}(w) \left(\int_0^{(1-x)/(1-w)} \frac{y^{n-1}}{1-y} dy \right) dw \\ &\quad + \frac{n}{2} \int_x^1 r_{n-1}(w) \left(\int_0^{(1+x)/(1+w)} \frac{y^{n-1}}{1-y} dy \right) dw, \qquad x \in [-1, 1] \end{aligned} \tag{A.11}$$

In particular, $r_n(-1)=r_n(1)=0$ for all $n \geqslant 1$. Calculating the derivative of $r_n(x)$ starting from (A.11), it is easily checked that for $n \geqslant 2$

$$\begin{aligned} r_n'(x) &= \frac{n}{2}\int_{-1}^{x} r_{n-1}(w)\left(\frac{d}{dx}\int_0^{(1-x)/(1-w)} \frac{y^{n-1}}{1-y}\,dy\right)dw \\ &\quad + \frac{n}{2}\int_x^1 r_{n-1}(w)\left(\frac{d}{dx}\int_0^{(1+x)/(1+w)} \frac{y^{n-1}}{1-y}\,dy\right)dw \\ &= -\frac{n}{2}\int_{-1}^{x} r_{n-1}(w)\left(\frac{d}{dw}\int_0^{(1-x)/(1-w)} \frac{y^{n-2}}{1-y}\,dy\right)dw \\ &\quad -\frac{n}{2}\int_x^1 r_{n-1}(w)\left(\frac{d}{dw}\int_0^{(1+x)/(1+w)} \frac{y^{n-2}}{1-y}\,dy\right)dw \\ &= \frac{n}{2}\int_{-1}^{x} r_{n-1}'(w)\left(\int_0^{(1-x)/(1-w)} \frac{y^{n-2}}{1-y}\,dy\right)dw \\ &\quad + \frac{n}{2}\int_x^1 r_{n-1}'(w)\left(\int_0^{(1+x)/(1+w)} \frac{y^{n-2}}{1-y}\,dy\right)dw \end{aligned} \tag{A.12}$$

In particular, $r_n'(-1)=r_n'(1)=0$ for all $n \geqslant 2$. Iterating (A.12), one can now show that $(d^k/du^k)\,r_n(-1)=0$ for $k=0, 1, 2,..., n-1$. The proof of (4.18) and thus the derivation of Section 4 are now complete.

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